AN INVARIANT SOLUTION OF THE TURBULENT BOUNDARY-LAYER EQUATIONS

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The problem of the group stratification of the system of equations describing motion in the laminar sublayer and the turbulent core is considered. The fundamental group admissible by the initial system is constructed; invariant solutions constructed on one of the subgroups lead to a system of ordinary differential equations. Joining of the solutions and interchange of the equations occur at the boundary of the laminar sublayer. A class of power-law flows of a turbulent boundary layer is investigated. In the region of decelerated motion a double-valued solution is found corresponding to attached or separated flow. The commonly used integral characteristics are calculated and presented in the form of an interpolation polynomial.

In turbulent boundary-layer calculations using the method of integral relations there arises the problem of choosing a family of velocity profiles with one or more independent parameters to determine integral thickness ($\delta^*$, $\delta^{**}$, etc.). It should be noted that so far there is no rational method of taking account of all profile shapes arising in cross sections of a turbulent boundary layer for a variable static pressure distribution. We describe below one possible method of establishing a family of velocity profiles for a turbulent boundary layer based on the use of a semiempirical theory of turbulence including the commonly used universal constants.

We use a two-layer scheme to calculate the turbulent boundary layer. According to this scheme motion in the laminar sublayer is described by equations of the form

$$v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 v_x}{\partial y^2};\quad \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0,$$

and in the turbulent core by

$$v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\frac{\partial}{\partial y}\left(\frac{\varepsilon}{\rho}\frac{\partial v_x}{\partial y}\right),$$

where $\varepsilon = \rho l^2 \left|\partial v_x/\partial y\right|$ is the coefficient of eddy viscosity and $l$ is the Prandtl mixing length; for simplicity we assume that $l$ = ky over the whole thickness of the boundary-layer, where k is the turbulence constant. Henceforth we use the notation

$$u=\frac{v_x}{u_\infty};\quad s=\mathrm{Re}_x;\quad \mathrm{Re}_x=\frac{\rho u_\infty x}{\nu};\quad \psi=\int u d\bar{y};$$

where $u_\infty$ is the characteristic velocity and $\bar{y} = \mathrm{Re}_y$.

Introducing the stream function $\psi$ and transforming to new variables s and $\psi$ we obtain the equations:

$$\frac{\partial z}{\partial s}=\sqrt{U_e^2-z}\,\frac{\partial^2 z}{\partial \psi^2};\qquad (1)$$

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in the turbulent core

$$\frac{\partial z}{\partial s} = \frac{k^2}{2} \sqrt{U_e^2 - z}\, \frac{\partial}{\partial \psi} \left\{ \bar{y}^2 \left| \frac{\partial z}{\partial \psi} \right| \frac{\partial z}{\partial \psi} \right\}, \tag{2}$$

where $U_e^2 - u^2 = z$ is the velocity defect and $U_e$ is the external inviscid flow field, where for generality we can set $U_e^2 = u_e^2(s) + 2\omega\psi$ , where $\omega$ is the ratio of the vorticity in the outer flow to the average vorticity in the turbulent boundary layer.

A suitable family of velocity profiles is established by solving Eqs. (1) and (2) and satisfying boundary conditions depending on the kind of problem considered.

To solve the problem formulated we apply the method of group stratification to Eqs. (1) and (2). This method, using the fundamental ideas of the theory of group properties of differential equations, is being more and more widely used at the present time as a result of a series of papers by V. Ovsyannikov [1-3] giving algorithms for the construction of Lie groups admissible by the system under consideration and the possibility of using these groups to find various classes of particular solutions of the initial system.

The Prandtl–Mises equation (4) for pure laminar flow is solved in [4] by the method of symmetric solutions in the region $P\{s > 0,\ 0 \leqslant \psi \leqslant \infty\}$, where the fundamental group admissible by Eq. (1) has a Lie algebra of operators with the following basis:

$$X_1 = \frac{\partial}{\partial s};\quad X_2 = \frac{\partial}{\partial \psi};$$

$$X_3 = s\frac{\partial}{\partial s} + \frac{m+1}{2}\psi\frac{\partial}{\partial \psi} + 2mz\frac{\partial}{\partial z} + \frac{3m-1}{2} w \frac{\partial}{\partial w}$$

for constraints of the form

$$u_e^2 = \beta s^{2m};\qquad \omega = \omega_0 s^{\frac{3m-1}{2}}.$$

Invariant solutions constructed on a subgroup corresponding to the operator $X_3$ and sought for in the form

$$\eta = \psi s^{-\frac{m+1}{2}},\quad z = Z(\eta)\, s^{2m},$$

lead to the ordinary differential equation

$$\sqrt{\beta + 2\omega_0\eta - Z}\cdot Z'' + \frac{m+1}{2}\eta Z' = 2mZ \tag{3}$$

where primes denote differentiation with respect to $\eta$.

Flow in the laminar sublayer was described by Eq. (3), and invariant solutions of Eq. (2) or the equivalent system Q describing flow in the turbulent core

$$\frac{\partial z}{\partial \psi} = w;\quad \frac{\partial \bar{y}}{\partial \psi} = \left(U_e^2 - z\right)^{-\frac{1}{2}};$$

$$\frac{\partial z}{\partial s} = \pm k^2\left(\bar{y}w^2 + \bar{y}^2 w \sqrt{U_e^2 - z}\, \frac{\partial w}{\partial \psi}\right),$$

were found as recommended in [1]. It was found that the fundamental group admissible by system Q is described by the Lie algebra of the operators

$$X_1 = \frac{\partial}{\partial s};\quad X_2 = \frac{\partial}{\partial \psi};$$

$$X_3 = s\frac{\partial}{\partial s} + (m+1)\psi\frac{\partial}{\partial \psi} + 2mz\frac{\partial}{\partial z} + (m-1) w\frac{\partial}{\partial w} + \bar{y}\frac{\partial}{\partial \bar{y}}.$$

Invariant solutions constructed on a subgroup with the operator $X_3$ were sought for in the form

$$\eta_1=\psi s^{-(m+1)};\quad z=Z_1(\eta_1)s^{2m};\ \overline{y}=Y(\eta_1)s;\ w=W(\eta_1)s^{m-1},$$

which after substitution into Q led to the following system of ordinary differential equations:

$$\pm k^2 Y\sqrt{\beta+2\omega_0\eta_1-Z_1}\left\{Y'Z_1'^2+YZ_1'Z_1''\right\}+(m+1)\eta_1 Z_1'=2mZ_1; \tag{4}$$

$$Y'=\frac{1}{\sqrt{\beta+2\omega_0\eta_1-Z_1}}. \tag{5}$$

To obtain a family of velocity profiles depending on a parameter it is necessary to integrate Eqs. (3), (4), and (5) subject to appropriate boundary conditions. For example, for boundary-value problems conditions of the form $Z(0)=\beta;\ Z_1(\infty)=0;\ Y(0)=0$ can be used. The interchange of equations in the use of the two-layer scheme occurs at the boundary of the laminar sublayer; in this case it is assumed [5] that in passing through the boundary of the sublayer the derivatives of the velocity have discontinuities $(\partial v_x/\partial y)_{y=\delta_l-0} = k_1(\partial v_x/\partial y)_{y=\delta_l+0}$ but the physical quantities themselves (velocity, friction pressure) are continuous. Using the latter considerations we write the equations which hold on the boundary of the laminar sublayer (characteristic values are denoted by an asterisk):
the physical conditions

$$\begin{gathered}(v_x)_{\delta_l-0}=(v_x)_{\delta_l+0},\quad \left(\frac{\partial v_x}{\partial y}\right)_{\delta_l-0}=k_1\left(\frac{\partial v_x}{\partial y}\right)_{\delta_l+0},\\ \delta_l=\delta_t,\ (\tau_{xy})_{\delta_l-0}=(\tau_{xy})_{\delta_l+0};\end{gathered} \tag{6}$$

the transformed conditions

$$\frac{Z^*}{\beta+2\omega_0\eta^*}=\frac{Z_1^*}{\beta+2\omega_0\eta_1^*},\quad \eta^*\left(2\omega_0-Z^{*\prime}\right)=k_1\eta_1^*\left(2\omega_0-Z_1^{*\prime}\right),$$

$$Y^*=\frac{\eta_1^*}{\eta^*}\int_0^{\eta^*}(\beta+2\omega_0\eta-Z)^{-1/2}\,d\eta,\quad 1=\frac{k^2}{2}Y^{*2}\left(\frac{\eta^*}{\eta_1^*}\right)^2\left|2\omega_0-Z_1^{*\prime}\right|.$$

The joining conditions determine the initial conditions for integrating system (4), (5). Obviously if the ordinates $\eta^*$ of the switching of Eq. (3) to Eqs. (4) and (5) were known it would be possible to determine from (6) the initial conditions $\eta_1^*,\ Z_1^*,\ Z_1^{*\prime}$, and $Y^*$. The missing equation must be obtained from the condition $\psi_{\delta_l-0}=\psi_{\delta_l+0}$. So far the problem is completely self-similar, since the coordinate s (or x) does not enter explicitly into either the initial equations or the boundary conditions. However, the requirement $\psi_l=\psi_T$ turns the problem into a locally self-similar problem since in reducing this condition to dimensionless form we obtain $\eta_1^*=\eta^*(\mathrm{Re}_x)^{(m+1)/2}$. This result is a consequence of the known lack of similarity for turbulent boundary-layers in view of the different character of the development of flow in the laminar and turbulent parts of the boundary layer. In the classical studies of Kármán [6] and Prandtl [7] it was assumed that such similarity exists; this assumption was based on measurements in a range of Reynolds numbers close to $\mathrm{Re}_x\sim 10^6-10^7$. Here the presence of local similarity is assumed and all the integral characteristics obtained by solving the problem were calculated for the Reynolds number $\mathrm{Re}_x=4\cdot 10^6$.

The parameter β appearing in Eqs. (3), (4), and (5) is not essential, since it can be eliminated from both the equations under consideration and the boundary conditions by a transformation of the form $\lambda = Z/\beta$, $\zeta=\eta\beta^{-1/4}$, $\lambda_1=Z_1/\beta$, $\zeta_1=\eta_1\beta^{1/2}$. In performing the calculations the system (3), (4), (5) was reduced to a system of five ordinary first-order differential equations by the substitutions $d\lambda/d\zeta=t$, $d\lambda_1/d\zeta_1=t_1$, and it turned out to be convenient as in [4] to use the variables t and $t_1$ and not η and $\eta_1$ as arguments. In this case the radical singularity appears in the numerator of the right-hand sides of the equations:

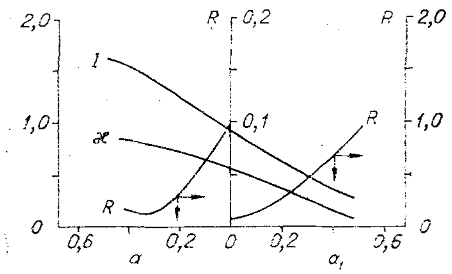


Fig. 1

$$\frac{d\zeta}{dt} = P, \quad \frac{d\lambda}{dt} = Pt, \tag{7}$$

$$\frac{d\zeta_1}{dt_1} = P_1, \quad \frac{d\lambda_1}{dt_1} = P_1 t_1, \quad \frac{dY}{dt_1} = \frac{P_1}{\sqrt{1 + 2\omega_0\zeta_1 - \lambda_1}}, \tag{8}$$

where

$$P = \frac{\sqrt{1 + 2\omega_0\zeta - \lambda}}{2m\lambda - \frac{m+1}{2}\zeta t}, \quad P_1 = \pm \frac{k^2 Y^2 t_1 \sqrt{1 + 2\omega_0\zeta_1 - \lambda_1}}{2m\lambda_1 - (m+1)\zeta_1 t_1 \mp k^2 Y^2 t_1^2}.$$

The boundary conditions have the form: at $t = t_0$ $\zeta = 0, \lambda = 1$, at $t = t_*$ $\zeta_1 = \zeta_R, \lambda_1 = 0$, where $\zeta_R$ corresponds to a reasonably chosen asymptotic value $\eta_1$ ( $\zeta_R$ = 10 was used in the calculations). Equations (7) and (8) were integrated by the Runge–Kutta method with refinements by Newton's method to satisfy the conditions at $t = t_*$. Specifying an arbitrary value of $t_0$ we integrate system (7), testing the joining conditions at each step; these are satisfied if

$$\frac{k^2}{k_1}\left\{\int_0^t (1 + 2\omega_0\zeta - \lambda)^{-1/2} P dt\right\}^2 \mathrm{Re}_x^{\frac{m+t}{2}} |2\omega_0 - \lambda'| = 2.$$

When (9) is satisfied Eqs. (6) determine the initial conditions for integrating system (8), which can be integrated so long as the conditon $\zeta_1 = \zeta_R$ is satisfied. At this point the condition $\lambda_1 = 0$ is tested, and if it is not satisfied the iteration process is repeated.

The system (7), (8) contains the two important parameters m and $\omega_0$. The range of m is determined from the condition for the existence of a solution of the original system of equations, and the range of $\omega_0$ is such that the basic premises of the concept of a boundary layer will not be violated. Most of the calculations were preformed for $\omega_0 = 0$, but the cases $\omega_0 = 0.5$ and 1.0 were considered. In view of the slight effect of $\omega_0$ on the integral characteristics of the boundary layer in the range considered these calculations are not presented here.

The pressure gradient parameter varies from m = 1 (rear stagnation point) to $m \simeq -0.286$, where the solution of the initial system still exists. In the range of $0.5 \leqslant m \leqslant 1$ the integral characteristics are practically independent of m; all the changes affect only the laminar sublayer which is small and in accelerated flow does not exert a significant effect on the structure of the turbulent core. Therefore, to increase the accuracy of the interpolation polynomials presented below the results of the calculations were processed up to m = 0.5.

In the region of decelerated flow m < 0, just as in the well known Hartree solution of the Falkner-Skan equation, a double-valued solution was found; for the same value of m < 0 there exist two independent solutions, one of which corresponds to attached flow at the surface, and the other to separated flow. Clauser [8] and Ludwieg and Tillmann [9] were the first to note this peculiarity.

The separation point of the turbulent boundary layer ($\tau_w = 0$) is characterized by the conditions $m_S = -0.2662$ and $H_S = (\delta^*/\delta^{**}) = 1.7566$. These results agree rather well with the experimental data of Stratford [10] who obtained $m_s \simeq -0.25$ and $H_S \simeq 1.8$-2.0. Figure 1 shows the calculated values of the shape parameters most frequently used in turbulent boundary-layer theory: $\varkappa = \delta^{**}/\delta^*$, $I = \delta^{***}/\delta^*$, and $R = 2\delta^*\int_0^\delta [(\partial/\partial y)(u/U_e)]^2\,dy$. The quantity R corresponds to the integral of the viscous dissipation and is used in the well-known Truckenbrodt method, and

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U_e}\right) dy, \quad \delta^{**} = \int_0^\delta \frac{u}{U_e}\left(1 - \frac{u}{U_e}\right) dy,$$

$$\delta^{***} = \int_0^\delta \frac{u}{U_e}\left(1 - \frac{u^2}{U_e^2}\right) dy.$$

In Fig. 1 the shape parameter $a \geqslant 0$ denotes the dimensionless velocity at the boundary of the laminar sublayer, and the shape parameter $a_1$, which decreases to the constant value $a_1^* = 0.035$, is the velocity along the dividing line of the flow $\psi = 0$, i.e., $a_1 = (u/U_e)_{\psi=0} = 0.035$. The latter condition follows from the fact that at $a = 0$ $(u/U_e)_{\psi=0} = 0.035$. At $a = 0$ the condition $a_1 = 0$ is satisfied. This creates certain advantages in using various integral methods to calculate separated turbulent flows.

The results of the calculations, processed by the method of least squares, can be presented as interpolation polynomials fo the following form:
attached and partially separated flow $\left(0 \leqslant a \leqslant 0.49;\ 0 \leqslant \left(\frac{u}{U_e}\right)_{\psi=0} \leqslant 0.035\right)$;

$$\varkappa = 0.5393 + 0.6301a + 0.4826a^2 + 0.6981a^3 - 3.4145a^4;$$
$$I = 0.9005 + 1.0113a + 1.1964a^2 + 7.1022a^3 - 17.0728a^4;$$
$$R = 0.1008 - 0.3854a - 0.0040a^2 + 1.6511a^3 - 1.3658a^4;$$

separated flow $(0 \leqslant a_1 \leqslant 0.51)$;

$$\varkappa = 0.5393 - 1.0766a_1 + 0.1011a_1^2 - 0.0618a_1^3 - 0.2131a_1^4;$$
$$I = 0.9005 - 2.1394a_1 + 3.1247a_1^2 - 6.0071a_1^3 + 5.9687a_1^4;$$
$$R = 0.1008 - 2.8594a_1 + 4.9339a_1^2 - 9.4434a_1^3 + 7.8162a_1^4.$$

The interpolation error does not exceed 2%.

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